MATHEMATICAL MODELING OF HEAT AND MOISTURE TRANSFER IN THE SEASONAL THAWING OF FROZEN SOILS

P. P. Permyakov, P. G. Romanov, and A. V. Stepanov

UDC 551.345:536.421.001.57

A mathematical model of heat and moisture transfer is proposed and results are presented from a numerical experiment.

In mastering the northern regions of the country, it is necessary to study heat and mass transfer processes in freezing—thawing soils. Whereas builders formerly considered only thermal conditions, this approach is now inadequate. The transport of moisture and salt in such soils must be taken into account along with thermal processes.

1. The salinization of soils, due to both natural causes and human activities, has a strong effect on physico-mechanical and mass-transfer properties. It was established as early as the 1940's that the freezing and melting of pore moisture occurs in a certain temperature range [1]. In recent years, many attempts [2, 3, etc.] have been made to experimentally and numerically establish the functional dependence of pore moisture on temperature. This is referred to as the function of the quantity of unfrozen water.

The effect of different salts on the equilibrium thermodynamic state of soil moisture was examined in [4-6]. With allowance for these findings, the mathematical model of combined heat and moisture transfer in frozen soils can be represented in the form

$$c\gamma \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) - c_w \rho_w v \frac{\partial T}{\partial x} + \varkappa \frac{\partial}{\partial \tau}\left(\frac{\gamma \omega_i}{1+\omega}\right), \tag{1}$$

$$\frac{\partial \omega_w}{\partial \tau} = \frac{\partial}{\partial x}\left(k \frac{\partial \omega_w}{\partial x}\right) + \frac{\partial}{\partial x}\left(k\delta \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial x}\left(D_0 \frac{\partial C_w}{\partial x}\right) - \frac{\partial \omega_i}{\partial \tau}, \tag{2}$$

$$\frac{\partial \omega_w C_w}{\partial \tau} = \frac{\partial}{\partial x}\left(D \frac{\partial C_w}{\partial x}\right) - v \frac{\partial C_w}{\partial x} - \frac{\partial \omega_i C_i}{\partial \tau}, \tag{3}$$

where

$$c = \frac{c_{sk} + c_w \omega_w + c_i \omega_i}{1+\omega}; \quad \gamma = (1+\omega)\rho;$$
$$\omega = \omega_i + \omega_w; \tag{4}$$
$$v = -k \frac{\partial \omega_w}{\partial x} - k\delta \frac{\partial T}{\partial x}.$$

This system was closed by using the equilibrium unfrozen-water function

$$\omega_w = \omega_{u.w}(T, \omega, C) \tag{5}$$

and the condition of freezing of the salt together with the soil moisture

$$C_i = k_1 C_w. \tag{6}$$

These two conditions distinguish the present formulation from those used in [7-11].

The boundary and initial conditions were assigned in the following form:

$$-\lambda \frac{\partial T}{\partial x} + c_w \rho_w v T = -\alpha(\tau)(T - T_e(\tau)) + c_w \rho_w q(\tau) T_e, \quad x = 0, \quad \tau > 0;$$

Institute of Physicotechnical Problems of the North, Yakutsk Branch of the Siberian Department of the Academy of Sciences of the USSR. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 57, No. 1, pp. 119-124, July, 1989. Original article submitted January 12, 1988.

 0022-0841/89/5701-0828$12.50

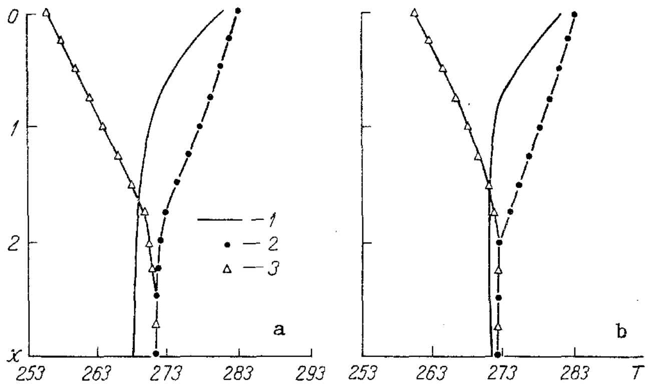

Fig. 1. Dynamics of the change in temperature through the depth of the sample: a) k_1 = 1; b) k_1 = 0.5; 1) end of May; 2) end of August; 3) end of December. x, m; T, K.

$$-k\frac{\partial\omega_{\text{w}}}{\partial x}-k\delta\frac{\partial T}{\partial x}-D_0\frac{\partial C_{\text{w}}}{\partial x}=q(\tau),\ x=0,\ \tau>0;$$

$$-D\frac{\partial C_{\text{w}}}{\partial x}+vC_{\text{w}}=0,\ x=0,\ \tau>0;$$

$$T=T^0,\ \omega=\omega^0,\ C=C^0,\ x=l,\ \tau>0;\ T(x,\ 0)=T^0(x),\ 0\leqslant x\leqslant l;$$

$$\omega(x,\ 0)=\omega^0(x),\ 0\leqslant x\leqslant l;\ C(x,\ 0)=C^0(x),\ 0\leqslant x\leqslant l.$$

2. System (1-3) is approximated at the internal nodes of the difference grid $\omega_{h\tau}$ = $\omega_\tau \times \omega_h$, where $\omega_\tau=\{j\cdot\Delta\tau,\ j=1,\ 2,\ \ldots\}$; $\omega_h=\{i\cdot h,\ i=\overline{0,\ N}\}$. Here, we use the following uniform monotonic difference scheme [12]:

$$c_{\text{ef}}T_{\bar\tau}=(\lambda T_{\bar x})_x-c_{\text{w}}\rho_{\text{w}}(v^-T_x+v^+T_{\bar x})+i\varkappa\rho\omega_{\bar\tau},$$

$$\omega_{\bar\tau}=(k\omega_{\text{w}\bar x})_x+(k\delta T_{\bar x})_x+(D_0(\nu S)_{\bar x})_x,$$

$$S_{\bar\tau}=(D(\nu S)_{\bar x})_x-(v^-(\nu S)_x+v^+(\nu S)_{\bar x}),$$

where

$$c_{\text{ef}}=c\gamma+\varkappa\frac{\partial\omega_{\text{u.w}}}{\partial T};\ i=i(T)=\omega_{\text{i}}/\omega;\ v=v^++v^-,$$

$$v^+=0{,}5(v+|v|)\geqslant 0,\ v^-=0{,}5(v-|v|)\leqslant 0;$$

$$S=\omega_{\text{w}}C_{\text{w}}+\omega_{\text{i}}C_{\text{i}},$$

$$C_{\text{w}}=\nu S,\ \nu=100/(k_1\omega_{\text{i}}+\omega_{\text{w}}). \qquad (7)$$

TABLE 1. Values of the Parameters of the Boundary Condition for the Soil

Parameter	Month											
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
T_e, K	231,5	238,4	251,1	265,4	278,1	287,3	290,8	287,1	278,7	265	245,9	234,7
α, W/(m^2·K)	0,65	0,56	0,56	0,8	13,6	20,3	20,8	19,8	13,8	3,87	1,67	0,85
Precipitation, mm	11	9	6	11	18	33	43	42	26	20	16	12
Form of precip., %:												
solid	100	100	98	87	19	—	—	—	7	83	100	100
liquid	—	—	—	4	59	97	100	100	82	5	—	—
mixed	—	—	2	9	22	3	—	—	11	12	—	—
Evaporation, mm	0,4	0,3	2,0	10	45,5	69	70	48	11	—2	0,4	0,4

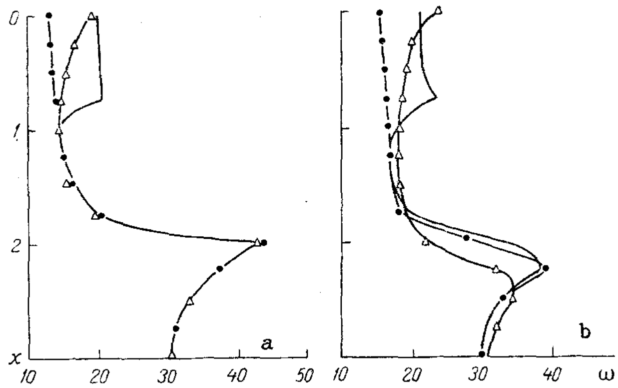

Fig. 2. Distribution of total moisture content at different moments of time. The notation is the same as in Fig. 1. ω, %.

We used the expressions for total moisture content ω (4) and salinity S (7) in deriving the difference equations for heat conduction and salinity. Writing the equations in this form improves the stability of the difference problems. The problems were solved numerically by the trial-run method with the use of iterations.

The content of unfrozen water is determined by equilibrium equation (5) with the concentration $\bar{C} = (C_w + C_i)/2$, i.e. it has the form $\omega_w = \omega_{u.w.}(T, \omega, \bar{C})$. At $k_1 \neq 1$ $(0 \le k_1 \le \infty)$ the concentrations of water C_w and ice C_i are unequal. We therefore take the averaged value. The parameters of the unfrozen-water function are found by analyzing experimental data from tests conducted with different concentrations in the mixture. These parameters are expressed as follows [3]:

$$\omega_{u.w} = \begin{cases} \omega_{s.w}, & T \leqslant T_1, \\ \dfrac{a}{|T-273|^{B}} + d, & T_1 \leqslant T \leqslant T_2, \\ (T-T_2)(\omega-\omega_e)/(T_3-T_2)+\omega_e, & T_2 \leqslant T \leqslant T_3, \\ \omega, & T_3 \leqslant T, \end{cases}$$

where a, B, d, ω_e, and $\omega_{s.w}$ are constants; $T_i = T_i(C)$ are functions of concentration. With a linear approximation, these functions have the form

$$T_i = \bar{T}_i + a_i\bar{C}, \quad i = 1, 2, 3.$$

Here, $\bar{T}_i$, a_i are empirical constants. With $a_i = 0$ $(i = 1, 2, 3)$, the usual equation of heat and mass transfer is obtained [13, 14].

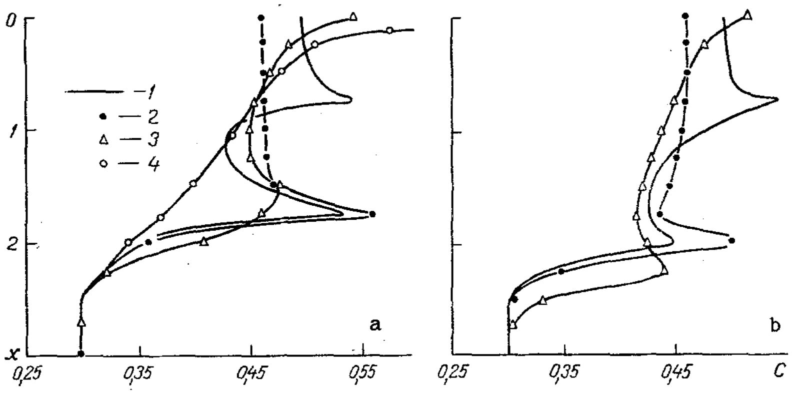

Fig. 3. Profile of the change in salt content: 1-3) see Fig. 1; 4) initial salt distribution (January of the first year) [19].

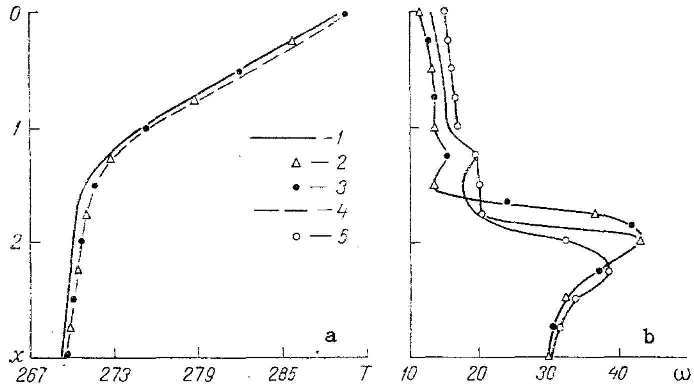

Fig. 4. Redistributions of temperature (a) and total moisture content (b) at the end of June obtained on the basis of different mathematical models: 1) heat and mass transfer with $k_1 = 1$; 2) with $a_i = 0$, $i = 1, 2, 3$; 3) heat and mass transfer; 4) heat transfer; 5) heat and mass transfer at $k_1 = 0.5$. T, K.

3. To compare the proposed model with other well-known models and to predict the dynamics of heat and mass transfer during the seasonal thawing of frozen soil, we performed a numerical experiment with application to the natural-climatic conditions in the Central Yakutsk region. We took the following initial values for the parameters in the numerical calculation [15]: $c_{sk} = 774$ J/(kg·K); $\rho = 1500$ kg/m^3; $L = 334$ kJ/kg;

$$\lambda(T, \omega, C) = 1{,}162\left(\lambda_f + (\lambda_t - \lambda_f)\,\frac{\omega_{u.w}(T, \omega, C) - \omega_{s.w}}{\omega - \omega_{s.w}}\right), \quad \text{W/(m·K)};$$

$$\lambda_{t,f} = m_{t,f}(0{,}001\rho + 0{,}1\omega - 1{,}1) - 0{,}1\omega;$$

$$k(T, \omega, C) = \begin{cases} k_0(T)\exp(k_2\omega_w - k_3\omega_i), & \omega_w \leqslant \omega_k; \\ k_0(T)\exp(k_2\omega_k - k_3\omega_i), & \omega_w \geqslant \omega_k; \end{cases}$$

$$m_t = 1{,}5;\ m_f = 1{,}7;\ k_0(T) = 1{,}5\cdot 10^{-8}(1 + a_0 T);$$

$$a_0 = \begin{cases} 0{,}004, & T \geqslant 273\ \text{K}, \\ 0, & T \leqslant 273\ \text{K}; \end{cases}$$

$$k_2 = 0{,}172;\ k_3 = 0{,}23;\ D = k\omega/200;\ D_0 = D;\ l = 10\ \text{m};\ h = 0{,}25\ \text{m};$$

$$\Delta\tau = 36{,}5\ \text{h};\ \overline{T}_1 = 266{,}1\ \text{K};\ \overline{T}_2 = 272{,}5\ \text{K};\ \overline{T}_3 = 273\ \text{K};$$

$$a_1 = a_2 = 0{,}6;\ a_3 = 0;\ a = 4{,}1;\ b = 0{,}73;\ d = 0;\ \delta = 0;$$

$$\omega_e = 6{,}8\%;\ \omega_{s.w} = 1\%;\ \omega_k = 30\%.$$

Table 1 shows the average monthly precipitation [16], evaporation [17], air temperature [18], and effective heat-transfer coefficient. The snow melts at the beginning of May and evaporation takes place in the summer months. It should be noted that evaporation is greater than precipitation in the mean annual balance.

Now let us examine the results of the numerical experiment. It can be seen from the temperature field (Figs. 1 and 4a) that salinity and the seepage factor affect the depth of the seasonal thawing and the rate of freezing of the soil.

Figures 2 and 4b show the distribution of total moisture content depthwise in a ground mass with allowance for infiltration and evaporation of precipitation. Snow water enters the soil in May and gradually penetrates to the seasonal thawing front. The reverse process is seen in the fall and winter – moisture is drawn upward toward the surface of the soil. Figure 3 shows the dynamics of the salt distribution according to the dry residue for different values of the seepage factor. At lower values of this factor, there is an increase in the concentration of unfrozen water and a corresponding expansion of the region of seasonal distribution of salt and total water. It can be seen from the figures that the salt distribution is closely related to the dynamics of moisture content. The character of the depthwise distribution of salt content is consistent with data obtained in field tests in [19, 20].

The points in Fig. 4 show the results obtained by means of heat and mass transfer models (1-2) without allowance for salt transport. The dashed curve shows the solution of temperature problem (1) at $v \equiv 0$. It is evident from the figure that the results agree well with each other even though the initial differential equations were approximated by different difference schemes.

Thus, the proposed mathematical model makes it possible to more accurately describe heat and mass transfer in thawing—freezing soils.

NOTATION

c, heat capacity of the soil; C, concentration of the solution, %; D, salt diffusion coefficient; D_0, osmotic diffusion coefficient; k, water diffusion coefficient; k_1, coefficient characterizing the freezing of the salts together with soil moisture; q, mass flux; T, temperature; T_1, T_2, T_3, temperature constants; T_e, temperature of the environment; S, salinity, % of the mass of the dry soil; ω, total moisture content, % (kg/kg); ω_w, ω_i, content of water in the liquid and solid state, %; $\omega_{u.w}$, content of unfrozen water, %; $\omega_{s.w}$, content of strongly bound water, %; x, space coordinate; α, heat-transfer coefficient; $\Delta\tau$, time step; δ, temperature gradient coefficient; $\varkappa$, latent heat of phase transformation of water referred to a unit mass of water; λ, thermal conductivity; τ, time; ρ, bulk density of the mineral skeleton. Indices: f, frozen; t, thawed; 0, initial; w, water; i, ice; ef, effective; sk, skeleton.

LITERATURE CITED

1. N. A. Tsytovich, Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz., 9, No. 5-6, 493-502 (1945).
2. E. D. Ershov, Moisture Transport and Cryogenic Textures in Dispersed Rock [in Russian], Moscow (1979).
3. A. R. Pavlov, P. P. Permyakov, and A. V. Stepanov, Inzh.-Fiz. Zh., 39, No. 2, 292-297 (1980).
4. L. I. Mahar, R. M. Wilson, and T. S. Vinson, Permafrost, 4th Int. Conf. Proc., Washington, July, 1983, pp. 773-778.
5. P. E. Patterson and M. N. Smith, ibid., pp. 968-972.
6. A. V. Stepanov, Development of Methods of Thermal Protection for Engineering Facilities in the Far North [in Russian], Yakutsk (1983), pp. 68-78.
7. N. N. Verigin, Hydrogeochemical and Physicochemical Properties of Rocks [in Russian], Moscow (1977).
8. A. M. Yakirevich, Tr. Vses. Nauchno. Issled. Inst. Gidrotekh. Melior., Sewage—Drainage Systems in the Arid Region [in Russian], Moscow (1986), pp. 76-85.
9. M. P. Galanin and N. A. Tikhonov, Modeling Soil Processes [in Russian], Pushchino (1985), pp. 76-82.
10. Ya. A. Pachevskii, E. V. Mironenko, et al., Tr. IPiF, Ecônomic Model, No. 8 (1982).
11. L. M. Reks, A. M. Yakirevich, and E. A. Zimina, Tr. Vses. Nauchno. Issled. Inst. Gidrotekh. Melior., Sewage—Drainage Systems in the Arid Region [in Russian], Moscow (1986), pp. 66-76.
12. N. V. Karetkina, Zh. Vychisl. Mat. Mat. Fiz., 20, No. 1, 236-240 (1980).
13. A. R. Pavlov and P. P. Permyakov, Inzh.-Fiz. Zh., 44, No. 2, 311-316 (1983).
14. Y. W. Jame and D. I. Norum, Water Resour. Res., 16, No. 4, 811-819 (1980).
15. A. V. Pavlov, Thermophysics of Terrains [in Russian], Novosibirsk (1979).
16. Handbook of Climate in the USSR, Vol. 24, Pt. IV, Moscow (1968).
17. A. R. Konstantinov, Evaporation in Nature [in Russian], Leningrad (1968).
18. M. K. Gavrilova, Climate of the Central Yakutsk Region [in Russian], Yakutsk (1973).
19. L. G. Elovskaya, D. K. Konorovskii, and D. D. Savvinov, Frozen Salt-Bearing Soils in Central Yakutsk [in Russian], Moscow (1966).
20. N. P. Anisimova, Formation of the Chemical Composition of Groundwater in Thawed Regions [in Russian], Moscow (1971).